Functions of a Complex Variable

If $z = x + iy$ and $w = u + iv$ are two complex variables and if for each value of z in a certain portion of the complex plane there corresponds one or more values of w, then w is said to be a function of z and is written as

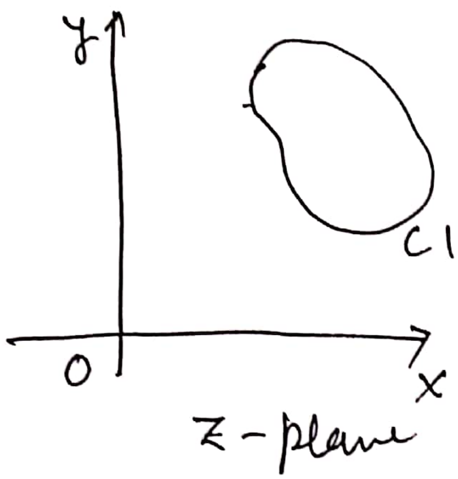
$$w = f(z) = f(x+iy) = u(x,y) + i\,v(x,y) \quad \text{— (1)}$$

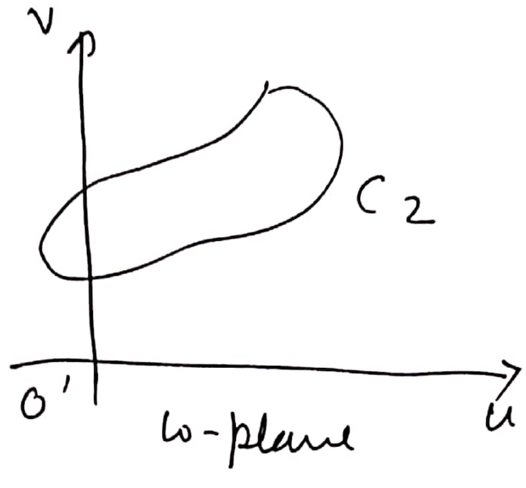
where $u(x,y)$ and $v(x,y)$ are real functions of the real variables x and y. Clearly for a given value of z, the values of x and y are known and thus, one or more values of w are determined by (1). If for each value of z in R, there corresponds only one value of w, then w is called a single-valued function of z.

We shall now consider the question of representing graphically the function of a complex variable. Real function of real variables, $y = f(x)$ can be exhibited graphically by plotting corresponding values of x and y as rectangular coordinates of points in the x-y plane. But in the complex domain, the functional relation

$$w = f(z) = u(x,y) + i\,v(x,y)$$

involves four real variables, namely, the two independent variables x and y and the two dependent variables u and v. Hence to plot $w = f(z)$ in the cartesian fashion, a space of four dimensions is necessary. This is not possible. Instead we make use of two complex Gauss planes, one for the variable $w = u+iv$. These are called the z-plane and the w-plane respectively. In the z-plane the point $x+iy$ is plotted and in the w-plane the point the point $u+iv$ is plotted. So in short we have two argand diagrams, one with the x and y variables and the ~~eigther wi~~ other with the u and v variables. A function $w = f(z)$ is now represented, not by locus of points is a four dimensional region but by a correspondence between the points of these two planes. corresponding to each point (x,y) in the z-plane for which $f(x+iy)$ is defined, there will be a point (u,v) in the w-plane, where $w = u+iv$





The function $w = f(z)$ thus define a mapping or transformation of the figures of the plane into the figures on the w-plane.

Limit and Continuity:

Let $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$, $z_0 = x_0 + iy_0$ and $L = u_0 + iv_0$. Then,

$\lim_{z \to z_0} f(z) = L$, if and only if

$\lim_{x \to x_0, y \to y_0} u(x,y) = u_0$ and $\lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0$.

Th If $\lim_{z \to z_0} f(z)$ exist, then it is unique

Ex $$\lim_{z \to 0} \frac{[\operatorname{Re} z - \operatorname{Im} z]^2}{|z|^2} = \lim_{z \to 0} \frac{(x-y)^2}{x^2+y^2}$$

consider any straight line path $y = mx$ (for different values of m we get different paths).

$$\lim_{x \to 0} \frac{(1-m)^2 x^2}{(1+m^2) x^2} = \frac{(1-m)^2}{1+m^2}$$

$\because$ the limit depends on m and is not unique, the limit does not exist

Continuity of a function

A function $f(z)$ is continuous at z_0, if the following three conditions are satisfied

(i) $f(z_0)$ exists (ii) $\lim_{z \to z_0} f(z)$ exists,

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$

Ex Show that the function $f(z)$ is not continuous at $z=0$, where

$$f(z) = \begin{cases} \dfrac{\text{Im}(z)}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

$$\lim_{z \to 0} \frac{\text{Im}(z)}{|z|} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y}{\sqrt{x^2+y^2}}$$

Consider the path $y = mx$. Then

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{y}{\sqrt{x^2+y^2}} = \lim_{x \to 0} \frac{mx}{\sqrt{1+m^2}\,x} = \frac{m}{\sqrt{1+m^2}}$$

which depends on m. Hence, the limit does not exist at $z=0$

$\therefore$ $f(z)$ is not cont^n at $z=0$

Differentiability

The function $f(z)$ is said to be differentiable at a point z_0, if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists.}$$

This limit is called the derivative of $f(z)$ at the point $z = z_0$ and is denoted by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{— (1)}$$

let $z - z_0 = \Delta z$ in (1)

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Remark

1. If $f(z)$ is differentiable at $z = z_0$, then it must be cont^n at $z = z_0$.

2. A function which is not cont^n at a point $z = z_0$ is not differentiable at $z = z_0$.

3. A function which is cont^n at a point $z = z_0$, may or may not be differentiable at $z = z_0$

4. The rules of differentiation of a function of a real variables x hold also for a function of Complex Var. z

Ex Show that the function $f(z)=\bar{z}$ is continuous at the point $z=0$ but not differentiable at $z=0$

Sol^n Let $z=x+iy$. Then $\bar{z}=x-iy$

$$f(z)=\bar{z}=x-iy$$

$$\Delta z=\Delta x+i\Delta y \qquad \Delta\bar{z}=\Delta x-i\Delta y$$

$$\lim_{z\to 0} f(z)=\lim_{\substack{x\to 0\\ y\to 0}}(x-iy)=0=f(0)$$

$\therefore$ $f(z)=\bar{z}$ is cont^n at $z=0$

How at $z=0$

$$\lim_{\Delta z\to 0}\frac{f(\Delta z)-f(0)}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left[\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}\right]$$

Consider now the path $y=mx$ $\quad \Delta y=m\Delta x$

$$\lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}=\lim_{\Delta x\to 0}\frac{\Delta x-im\Delta x}{\Delta x+im\Delta x}=\frac{1-im}{1+im}$$

which depends on m. Thus, the limit doesnt exist.

$\therefore$ $f(z)=\bar{z}$ is not differentiable at $z=0$

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Ex Show that the function $f(z)=\bar{z}$ is continuous at the point $z=0$ but not differentiable at $z=0$

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$$f(z)=\bar{z}=x-iy$$

$$\Delta z=\Delta x+i\Delta y \qquad \Delta\bar{z}=\Delta x-i\Delta y$$

$$\lim_{z\to 0} f(z)=\lim_{\substack{x\to 0\\ y\to 0}}(x-iy)=0=f(0)$$

$\therefore$ $f(z)=\bar{z}$ is cont^n at $z=0$

Now at $z=0$

$$\lim_{\Delta z\to 0}\frac{f(\Delta z)-f(0)}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left[\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}\right]$$

Consider now the path $y=mx$ $\quad \Delta y=m\Delta x$

$$\lim_{\Delta z\to 0}\frac{\overline{\Delta z}}{\Delta z}=\lim_{\Delta x\to 0}\frac{\Delta x-im\Delta x}{\Delta x+im\Delta x}=\frac{1-im}{1+im}$$

which depends on m. Thus, the limit doesnt exist.

$\therefore$ $f(z)=\bar{z}$ is not differentiable at $z=0$

Ex Show that the function $f(z) = |z|^2$ is differentiable only at $z = 0$ and no where else.

Sol^n $f(z) = |z|^2 = z\bar{z}$

$$\therefore \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[\frac{\cancel{z\bar{z}} + z\overline{\Delta z} + \Delta z \cdot \bar{z} + \Delta z \overline{\Delta z} - \cancel{z\bar{z}}}{\Delta z} \right]$$

$$= \lim_{\Delta z \to 0} \left[z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z} \right] = 0 \text{ for } z = 0$$

$\Rightarrow$ $f(z) = |z|^2$ is differentiable at $z = 0$

and $f'(0) = 0$

for $z \neq 0$

$$\lim_{\Delta z \to 0} \bar{z} + \overline{\Delta z} = \bar{z} \quad \text{but } \frac{\overline{\Delta z}}{\Delta z} \text{ does not exist.}$$

$\Rightarrow$ $|z|^2$ is not differentiable at $z \neq 0$

Analytic Function

A function $f(z)$ of a complex variable z is said to be analytic at a point z_0. Thus analyticity at a point z_0 means differentiability in some open disk about z_0.

Note Analyticity implies differentiability but not vice versa.

eg (i) $f(z) = |z|^2$ is differentiable only at $z=0$ and no where else. Therefore, the function is differentiable at $z=0$ but not analytic anywhere.

(ii) $f(z) = z^n$, n +ve integer, is diff at all points and therefore is analytic at every point of the finite complex plane.

Remark A function $f(z)$ which which is analytic at every point of the finite complex plane is called an entire function.

Cauchy-Riemann Equation

Th Suppose that the function $f(z) = u(x,y) + iv(x,y)$ is differentiable at z then at this point the first-order partial derivatives of u and v exist and satisfy the CR-equation.

$$u_x = v_y \qquad u_y = -v_x.$$

Remark 1. C.R equations are the necessary conditions for differentiability and analyticity of the function $f(z)$ at a given point.

i.e If f is analytic ($\Rightarrow$ differentiable) in a domain D, then u, v satisfy C-R eqn. at all points of D.

2 C.R eqn is not sufficient condition for analyticity

i.e If CR-eqn is satisfied $\not\Rightarrow$ fun is analytic.

3

Ex Show that the function

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0 & , z = 0 \end{cases}$$

Satisfies the C-R eqn at $z=0$ but $f'(0)$ doesnt exist.

Solⁿ $f(z) = u(x,y) + iv(x,y)$

$$u(x,y) = \frac{x^3-y^3}{x^2+y^2} \qquad v(x,y) = \frac{x^3+y^3}{x^2+y^2}$$

$(x,y) \neq (0,0)$

$\because f(0) = 0 \Rightarrow u(0,0) = 0, \; v(0,0) = 0$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h} \quad \text{at } z=0$$

$$= \lim_{h\to 0} \frac{h^3/h^2 - 0}{h} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(x,y+k) - u(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{u(0,k) - u(0,0)}{k} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(x+h, y) - v(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{v(0+h, 0) - v(0,0)}{h} \quad \text{at } z=0$$

$$= \lim_{h\to 0} \frac{h^3/h^2}{h} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(x, y+k) - v(x,y)}{k}$$

$$= 1 \quad \text{at } z=0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\boxed{u_x = v_y} \qquad \boxed{u_y = -v_x}$$

Thus CR eqns are satisfied at $z=0$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{x^3 - y^3 + i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

$$= \lim_{z \to 0} \frac{(1+i)(x^3+iy^3)}{(x^2+y^2)(x+iy)}$$

$$= \lim_{z\to 0} \frac{(1+i)(x^3+iy^3)(x-iy)}{(x^2+y^2)(x+iy)(x-iy)} = \lim_{z\to 0} \frac{(1+i)(x^3+iy^3)(x-iy)}{(x^2+y^2)^2}$$

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(13)

Choosing the path $y = mx$, we get-

$$\lim_{x \to 0} \frac{(1+i)(1+im^3)(1-im)\,x^4}{(1+m^2)^2\,x^4}$$

$$= \frac{(1+i)(1+im^3)(1-im)}{(1+m^2)^2}$$

which depends on m

$\Rightarrow$ The limit doesn't exist

$\Rightarrow$ $f'(0)$ doesn't exist

<u>Sufficient Conditions for a function to be analytic</u>

<u>Th</u> Suppose that the real and imaginary parts $u(x,y)$ and $v(x,y)$ of the function $f(z) = u(x,y) + iv(x,y)$ are continuous and have continuous first order partial derivatives in a domain D. If u and v satisfy the C-R eqn at all points in D, then the function $f(z)$ is analytic in D and

$$f'(z) = u_x + iv_x = v_y - iu_y$$

<u>Remark</u> f analytic $\Rightarrow$ C R eqn satisfied

$+$

f analytic. $\Leftarrow$ cont^n Partial Derivative e.g $f(z) = |z|^2$ is not analytic at any point

$f(z) = |z|^2 = \cancel{|x+iy|^2 = x-i}$

$= z\bar{z}$

$= (x+iy)(x-iy) = x^2+y^2$

$u(x,y) = x^2+y^2 \qquad v(x,y) = 0$

$\Rightarrow$ u and v are contn.

$u_x = 2x \qquad u_y = 2y \qquad v_x = 0 \qquad v_y = 0.$

$\Rightarrow$ partial derivatives are cont

C R eqn is satisfied only when

$u_x = v_y \qquad u_y = -v_x$

$\Rightarrow \quad 2x = 0 \qquad 2y = 0$

$\Rightarrow x = 0, \; y = 0$

$\Rightarrow$ fun is diff only at $(0,0)$

$\Rightarrow$ fun is not analytic at any point.

Ex 19 (Chandrika Prasad)

2. If $f(z) = x^2 + iy^2$, does $f'(z)$ exist at any point

Sol^n $f'(z)$ exist if first order Partial derivative is cont^n and satisfy CR eqn.

$f(z) = u + iv = x^2 + iy^2$

$u(x,y) = x^2 \qquad v(x,y) = y^2$

$u_x = 2x \qquad u_y = 0 \qquad v_y = 2y \qquad v_{\cancel{x}} = 0$

$\Rightarrow$ First order Partial Derivative are cont^n.

CR eqn is satisfied when

$u_x = v_y \Rightarrow 2x = 2y \Rightarrow x = y$

Hence $f'(z)$ exist at all points on the line $y = x$.

6. Verify whether the fun^n

$$f(z) = \frac{x^3y(y-ix)}{x^6+y^2} \quad \text{for } z \neq 0$$
$$= 0 \quad \text{for } z = 0$$

is analytic at $z=0$

Sol^n $f'(z) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{\frac{x^3y(y-ix)}{x^6+y^2} - 0}{x+iy}$

$$= \lim_{z\to 0} \frac{x^3y(x-iy)(y-ix)}{(x^6+y^2)(x+iy)(x-iy)} = \lim_{z\to 0} \frac{x^3y(x-iy)(y-ix)}{(x^6+y^2)(x^2+y^2)}$$

along $y = mx^3$

$$= \lim_{z\to 0} \frac{x^3mx^3(x-im^3)(mx^3-ix)}{(x^6+m^2x^6)(x^2+m^2x^6)}$$

$$= \lim_{x\to 0} \frac{\cancel{x^8}m(1-imx^2)(mx^2-i)}{\cancel{x^8}(1+m^2)(1+m^2x^4)}$$

$$= \lim_{x\to 0} \frac{-mi(1-i\cancel{m}x^2)(1+i\cancel{m}x^2)}{(1+m^2)(1+\cancel{m^2x^4})}$$

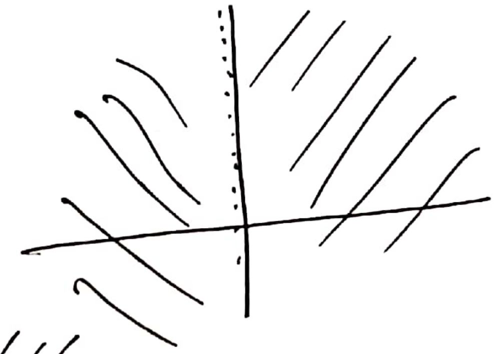
$$= \frac{-mi}{1+m^2}$$

$f'(z)$ doesnt exist at $z=0$ $\Rightarrow$ f is not analytic at $z=0$.

Connected Set: A set is connected if any two points z_1 and z_2 belonging to S can be joined by a polygon line ε [a path which consists of finetely many straight line segments) which is totally contained in S.

(a) $S = \{ z : Re(z) \neq 0 \}$

not connected



(b) $S = \{ z : Im(z) > 1 \}$

Connected

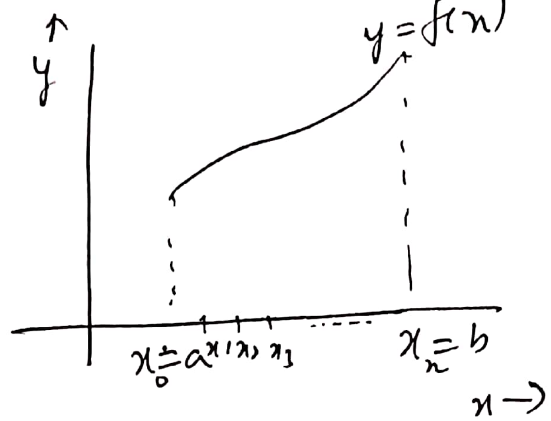
Integration of Complex Functions

Integration of function of a complex variable plays a very important role in many areas of science and engineering. Complex integration approach can be used to evaluate many improper integrals of a real variable, which cannot be evaluated using real integral calculus.

Line Integral in the Complex plane

In real case $\int_a^b f(x)\,dx$ where x is real and $f(x)$ is contn is the limit of the sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

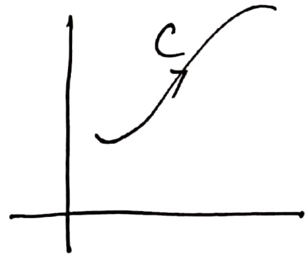


If such partition exist then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

Δx_i is the length of the interval. We want to extend this defn in the complex plane.

Let us first define the defn of curves in the complex plane.



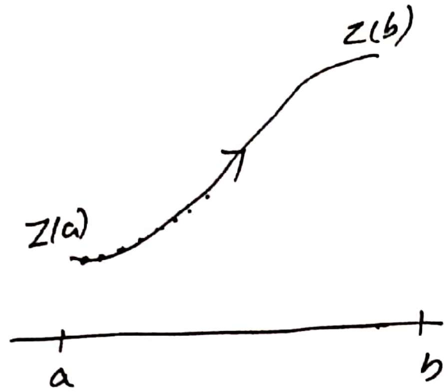
Defn (Curves in the Complex Plane)

Let $x(t)$ and $y(t)$ be two contn function of a real variable t, $a \le t \le b$

If $Z(t) = x(t) + i\,y(t)$ where $a \le t \le b$ then the point $z(t)$ will trace the curve as t varies from a to b with initial point $z(a)$ and terminal point $z(b)$

So the eqn of the curve C in the parametric form is $C: z(t) = x(t) + i\,y(t)$, $a \le t \le b$

As t varies $z(t)$ varies and it traces the curve.



Direction of the curve C, as t increases is taken as the +ve direction and opposite to it will be -ve direction

If the initial and terminal points coincide then it is closed curve

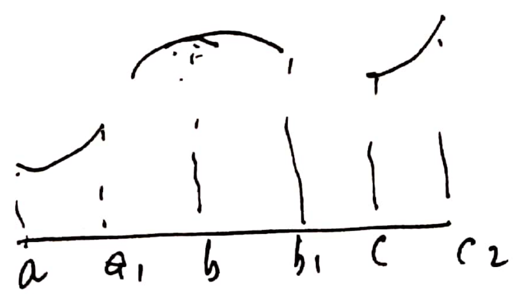
i.e a curve $z(t) = x(t) + iy(t)$ is said to be closed if $z(a) = z(b)$

A curve is said to be simple if it doesnt cross or intersect except at the point a and b

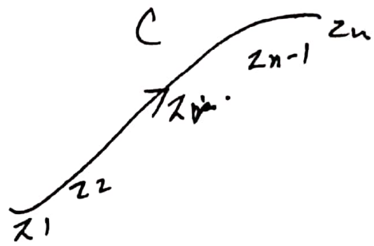
i.e $z(t_1) \neq z(t_2)$, $t_1 \neq t_2$

Note 1. If $z(t) = x(t) + iy(t)$ is cont^n then $x(t)$ and $y(t)$ are both cont^n and vice versa.

2. A curve $z(t) = x(t) + iy(t)$ is a piecewise cont^n if it consists of finite number of cont^n arc over $[a, b]$



Line Integral let C be any cont^n curve in the z plane. Divide C into n parts by the points $z_1, z_2, \ldots z_{n-1}, z_n$ as in the figure, and put



$\Delta z_r = z_r - z_{r-1}$ and limit of the sum

$$\sum_{r=1}^{n} f(z_r)\,\Delta z_r$$ as $r \to \infty$ and each $\Delta z_r \to 0$ if it exist, is know as the integral of $f(z)$ over the contour C.

ie $$\int_C f(z)\,dz = \lim_{r\to\infty} \sum_{r=1}^{n} f(z_r)\,\Delta z_r$$

Then $\int_C f(z)\,dz$ is known as line integral of $f(z)$ along the path C.

Note If C is closed then we denote it as

$$\oint_C f(z)\,dz.$$

Exercise 21

1. Evaluate $\displaystyle\int_{1-i}^{2+i} (2x+2iy+3)\,dz$

(i) along the path $x=t+1$, $y=2t^2-1$;

(ii) along the straight line joining $1-i$ to $2+i$

Sol^n (i) $x=t+1 \quad y=2t^2-1$

$$dz = dx+i\,dy = (dt+4t\,i\,dt) = (1+4it)\,dt$$

$$\int_{1-i}^{2+i}(2x+2iy+3)\,dz = \int_0^1 \left[2(t+1)+2i(2t^2-1)+3\right] \cancel{dx+}\,(1+4it)\,dt$$

$\because x=t+1,\ y=2t^2-1$, $2t^2=y+1$

$t=x-1$

$x=1 \quad t=0$

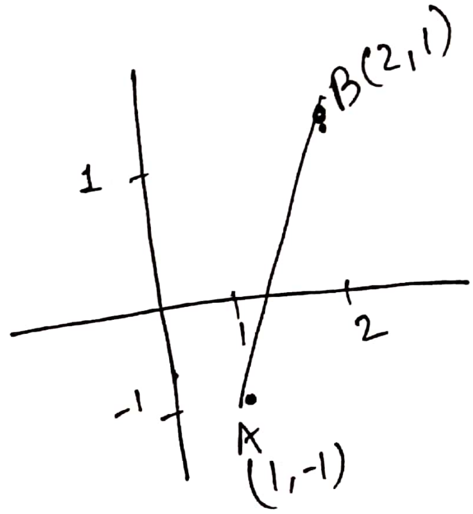
$x=2 \quad t=1$

$$=\int_0^1 -16t^2+12it^2+(10+20i)t+(5-2i)\,dt$$

$$=\left[\frac{-16t^4}{4}+\frac{12it^3}{3}+(10+20i)\frac{t^2}{2}+(5-2i)t\right]_0^1$$

$$= 6+12i$$

(ii) eqn of line AB



$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y+1}{x-1}=\frac{1+1}{2-1} \Rightarrow (y+1)=2(x-1)$$

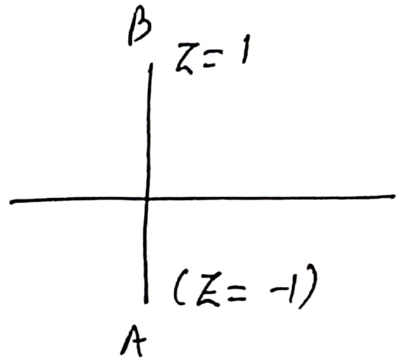
$$y=2x-3$$

Q3 Evaluate $\int_C |z|\,dz$ where C is the contour

(i) the straight line $z=-i$ to $z=i$

(ii) left half of the circle $|z|=1$ from $z=-i$ to $z=i$

(i) On the straight line from $z=-i$ to $z=i$

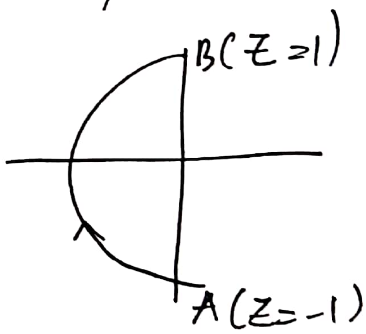


$x=0$

$z=iy \quad dz=i\,dy \quad |z|=|y|$

$$\int_C |z|\,dz = \int_{-1}^{1} |y|\,i\,dy = 2i\int_0^1 y\,dy = i$$

(ii) On left half of unit circle $|z|=1$ from $z=-i$ to $z=i$



$z=e^{i\theta}$ $\quad\theta$ varies from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$

$|z|=1 \quad dz = ie^{i\theta}d\theta$

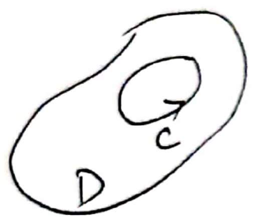
$$\int_C |z|\,dz = \int_{-\pi/2}^{\pi/2} ie^{i\theta}d\theta = \left[e^{i\theta}\right]_{-\pi/2}^{\pi/2} = \cancel{2}i$$

$$= \left[e^{i\pi/2} - e^{-i\pi/2}\right] = \left\{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - i\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right\}$$

$$= 2i$$

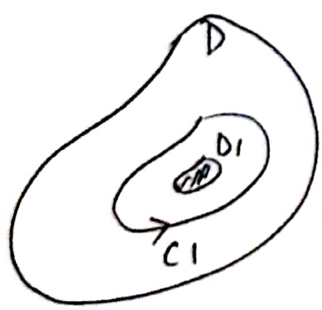
<u>Simply Connected Domain</u>: A connected domain is simply connected if every simple closed curve inside D encloses only points of D, or any closed curve which lies in D can be shrunk to a point inside D without leaving D

<u>e.g</u> A plane sheet of paper with no holes in it is a simply connected. If any closed curve C is shrunk, then it shrunks to a point lying inside D



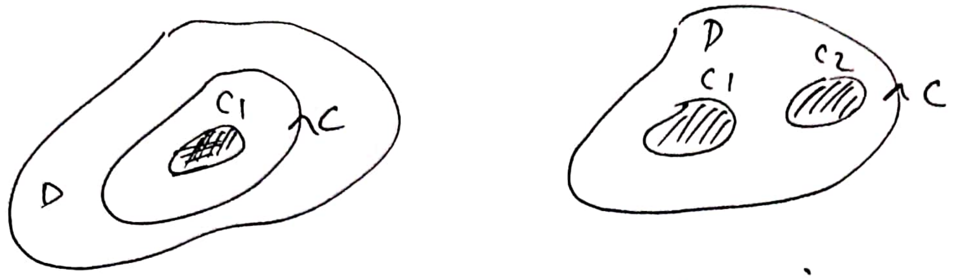
<u>Multiply Connected domain</u>: A domain which is not simply connected.

<u>e.g</u> 1. A plain sheet of paper from which some interior parts are removed is multiply connected domain.

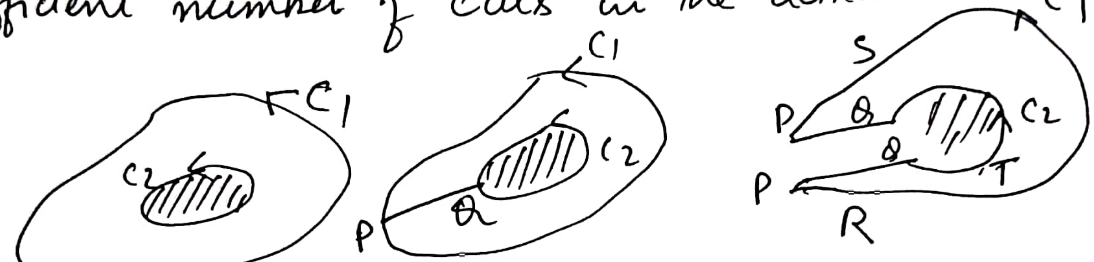


In this figure, let D_1 be removed from D. If a closed curve C_1 is shrunk, then it cannot be shrunk to a point as D_1 is not a part of D

2. Annulus $r < |z| < R$ because any closed curve C between the circles $|z| = R$ and $|z| = r$ cannot be shrunk to a point. Thus we may say multiply connected domain has holes



Any multiply connected domain can be connected into a simply connected by introducing sufficient number of cuts in the domain



The new domain PRSPQTQP

<u>Cauchy integral Theorem</u>: Let $f(z)$ be analytic and $f'(z)$ be continuous in a simply connected domain D, then $\oint_C f(z)\,dz = 0$ along any closed curve C contained in D.

Cauchy integral theorem uses two conditions, (i) the domain is simply connected and (ii) f(z) is analytic and f'(z) is continuous in D. The condition that D is simply connected is a necessary condition. The theorem cannot be applied for multiply connected domain. However, the condition that f(z) is analytic and f'(z) is cont^n in D are sufficient

$$\oint \frac{dz}{(z-z_0)^n} = 0 \quad n \neq 1 \quad \text{where } C: |z-z_0| = r$$

Here at $z = z_0$ $f(z) = \frac{1}{(z-z_0)^n}$ is not diff at $z = z_0$ hence not analytic at $z = z_0$.

Q Evaluate the integral $I = \int z^n dz$, $n = 0, \pm 1, \pm 2, \ldots$ where $C: |z| = r$ is traversed in the counter clockwise direction

Sol^n $|z| = r$ in the parametric form

$$z(t) = r(\cos t + i \sin t) \qquad 0 \le t \le 2\pi$$

$$z'(t) = r(-\sin t + i \cos t)$$

$$f(z(t)) = [z(t)]^n = r^n(\cos nt + i \sin nt)$$

$$I = \int_C f(z)\,dz = \int_0^{2\pi} f(z(t))\, z'(t)\, dt$$

$$= \int_0^{2\pi} r^{n+1} (\cos nt + i\sin nt)(-\sin t + i\cos t)\, dt$$

$$= r^{n+1} \int_0^{2\pi} [- \sin(n+1)t + i\cos(n+1)t]\, dt \quad , n \neq -1$$

$$= \frac{r^{n+1}}{n+1} \left[\cos(n+1)t + i\sin(n+1)t\right]_0^{2\pi}$$

$$= 0$$

For $n = -1$

$$I = \int_0^{2\pi} i\,dt = 2\pi i$$

$$\int_C z^n dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1 \end{cases}$$

Similarly $I = \int_C (z - z_0)^n dz \quad , n = 0, \pm 1, \pm 2 \ldots$

where $|z - z_0| = r \qquad z - z_0 = w$

$$I = \int_C w^n dw \qquad C = |w| = r$$

$$\int_C (z - z_0)^n dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1 \end{cases}$$

Corollary: If $f(z)$ is analytic in D and if two points A and B in D are joined by two curves C_1 and C_2 lying wholly in D then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

ie $\int f(z)\,dz$ is independent of the path C and depends only on the end points z_1 and z_2

Ex (i) $I = \oint_C e^{\sin z^2}\,dz$; $|z| = 1$

$f(z)$ is analytic, $f(z)$ is continuous

By Cauchy Integral theorem $I = 0$

(ii) $\cancel{\oint \sin z}$ $I = \oint_C \tan z\,dz$ $C: |z| = 1$

$\tan z$ is analytic except at the points $\cos z = 0$

or $z = \pm n\frac{\pi}{2} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

which lies outside C

$\Rightarrow I = 0$

Q Evaluate the integral $I = \oint_C \frac{3z+5}{z^2+2z}\,dz$, $C: |z|=1$

we have $f(z) = \frac{3z+5}{z^2+2z} = \frac{3z+5}{z(z+2)} = \frac{1}{2}\left[\frac{5}{z} + \frac{1}{z+2}\right]$

$$I = \oint_C \frac{3z+5}{z^2+2z}\,dz = \frac{5}{2}\oint_C \frac{dz}{z} + \frac{1}{2}\oint_C \frac{1}{z+2}\,dz$$

$$= I_1 + I_2$$

In the first integral I_1, the integrand is not analytic at $z=0$ which lies inside C.

$\therefore$ Cauchy integral theorem cannot be applied.

However, using the result

$$I_1 = \frac{5}{2}(2\pi i) = 5\pi i \qquad \oint_C z^n\,dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

In the second integral I_2, the integrand is analytic everywhere except at $z=-2$, which lies outside the circle $|z|=1$

$\therefore$ By the Cauchy integral theorem, we get $I_2 = 0$

$$I = I_1 + I_2 = 5\pi i$$

or $\oint_{C_1} f(z)dz + \int_{PQ} f(z)dz + \oint_{C_2^*} f(z)dz - \int_{PQ} f(z)dz = 0$

$$\Rightarrow \oint_{C_1} f(z)dz = -\oint_{C_2^*} f(z)dz = \oint_{C_2} f(z)dz.$$

where C_1 and C_2 are traversed in the same direction.

This result can be generalized to multiply connected domains. Let $f(z)$ be analytic in a domain D bounded by non ~~decreasing~~ intersecting simple closed curves $C, C_1, C_2, \ldots, C_n$ where $C_1, C_2, \ldots C_n$ lie inside C. Let $f(z)$ be analytic on C and C_i. We introduce n cuts and convert D to a simply connected domain. Applying the Cauchy integral theorem, we get

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_n} f(z)dz.$$

where all the curves are traversed in the positive direction.