Unit 3:-

Topics: Basic theory of Laplace Transform, Laplace Transform of elementary functions, Inverse Laplace Transform

Problems:

- Example 8.1-8.5 (on page 8.2 & 8.3) (J&I)
- Q. 1- 10/Exercise 8.1 (on page 8.6) (J&I)
- Q. 19—22/Exercise 8.1 (on page 8.6) (J&I)
- Q. 31—40/Exercise 8.1 (on page 8.6 & 8.7) (J&I)

Topics: Laplace Transform of derivatives, Laplace Transform of Integrals, Translation Theorem (First Shifting Property)

Problems:

- Example 8.16, 8.17 & 8.22 (on page 8.11, 8.12 & 8.15) (J&I)
- Q. 1—19/Exercise 8.2 (on page 8.13) (J&I)
- Q. 1—21/Exercise 8.3 (on page 8.22) (J&I)

Topics: Differentiation of Laplace Transform, Integration of Laplace Transform, Unit Step Functions, Translation Theorem (Second Shifting Property)

Problems:

- Example 8.32, 8.33 & 8.37 (on page 8.29 & 8.33) (J&I)
- Q. 10—22/Exercise 8.4 (on page 8.38) (J&I)
- Q. 32—42/Exercise 8.4 (on page 8.38) (J&I)
- Q. 29—52/Exercise 8.3 (on page 8.23) (J&I)

Topics: Periodic Functions, Convolution Theorem

Problems:

- Q. 2, 3, 6, 8 /Exercise 8.5 (on page 8.43) (J&I)
- Q. 46—55/Exercise 8.4 (on page 8.38) (J&I)

Topic: Applications to linear differential equations (Initial Value Problems)

Problems:

- Q. 23—36/Exercise 8.2 (on page 8.13) (J&I)
- Q. 22—28/Exercise 8.3 (on page 8.22)
- Q. 23—31/Exercise 8.4 (on page 8.38) & Q. 58—64/Exercise 8.4 (on page 8.39) (J&I)
- Q. 53—65/Exercise 8.3 (on page 8.24) (J&I)
- Q. 14—20/Exercise 8.5 (on page 8.44) (J&I)

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Course Incharge

Example 8.1 Find the Laplace transform of $f(t) = 1,\ t \geq 0$.

Solution From the definition, we have

$$\mathcal{L}[1] = \int_0^\infty e^{-st} \cdot 1\, dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s},\ s > 0.$$

Example 8.2 Find the Laplace transform of $f(t) = t,\ t \geq 0$.

Solution We have

$$\mathcal{L}[t] = \int_0^\infty e^{-st} \cdot t\,dt = \left[-\frac{e^{-st}t}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2},\ s > 0.$$

Example 8.3 Find the Laplace transform of $f(t) = e^{at},\ t \geq 0$.

Solution We have

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a},\ s > a.$$

Example 8.4 Find the Laplace transform of $f(t) = \sinh \omega t,\ t \geq 0$.

Solution We write $\sinh \omega t = (e^{\omega t} - e^{-\omega t})/2$. Using the linearity principle, we obtain

$$\mathcal{L}[\sinh \omega t] = \frac{1}{2}\int_0^\infty e^{-st}(e^{\omega t} - e^{-\omega t})\, dt = \frac{1}{2}\int_0^\infty [e^{-(s-\omega)t} - e^{-(s+\omega)t}]dt$$

$$= \frac{1}{2}\left[-\frac{e^{-(s-\omega)t}}{s-\omega} + \frac{e^{-(s+\omega)t}}{s+\omega}\right]_0^\infty = \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right]$$

$$= \frac{\omega}{s^2 - \omega^2},\ s > \omega.$$

Example 8.5 Find the Laplace transform of $f(t) = \cos at,\ t \geq 0$.

Solution We write $\cos at = \text{Re}\,(e^{iat})$ and consider $\mathcal{L}[e^{iat}]$.
We have

$$\mathcal{L}[e^{iat}] = \mathcal{L}[\cos at + i \sin at] = \mathcal{L}[\cos at] + i\,\mathcal{L}[\sin at].$$

Now, from Example 8.3

$$\mathcal{L}[e^{iat}] = \frac{1}{s - ia} = \frac{s + ia}{s^2 + a^2}.$$

Therefore, comparing the real parts on both sides, we obtain

$$\mathcal{L}[\cos at] = \frac{s}{s^2 + a^2}.$$

Exercise 8.1

Using the definition, find the Laplace transform of the following functions.

1. $2t - 5$.
2. $at^2 + bt + c$.
3. $(2 + 3t)^2$.
4. $\sin(3t + 2)$.
5. $\cos(at + b)$.
6. $\sin^2(\omega t)$.
7. $\sinh^2 t$.
8. $\cosh^2 at$.
9. $\cos t - \sin t$.
10. $3t^2 - \cos 2t$.
11. te^{2t}.
12. e^{-2t+5}.
13. t^2e^t.
14. $t \sin t$.
15. $t \cos 2t$.
16. $e^t \cos t$.
17. $e^{-t} \sin t$.
18. $(\cos t + \sin t)^2$.
19. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & t \ge 1, \end{cases}$
20. $f(t) = \begin{cases} 2, & 0 \le t < 3, \\ 0, & t \ge 3. \end{cases}$
21. $f(t) = \begin{cases} \cos t, & 0 \le t < \pi, \\ 0, & t \ge \pi. \end{cases}$
22. $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ \sin t, & t \ge \pi. \end{cases}$

Find the Laplace transform of the following functions.

23. $(t + 3)^2$.
24. $6 - e^{3t}$.
25. $3t^2 - 5e^{-2t} + 6$.
26. $\sin 5t + \cos 4t$.
27. $\sinh 2t + \cosh 2t$.
28. $e^{-t} \sinh t$.
29. $e^t \cosh t$.
30. $e^t \sin t$.

In the following problems, the Laplace transform $F(s) = \mathcal{L}[f(t)]$ is given. Find the inverse Laplace transform $f(t)$.

31. $\dfrac{3}{s+5}$.
32. $\dfrac{\pi}{s^2 + \pi^2}$.
33. $\dfrac{s}{s^2 - 4}$.
34. $\dfrac{6}{s^4}$.
35. $\dfrac{s+3}{(s-1)(s+2)}$.
36. $\dfrac{2}{s^3} + \dfrac{6}{s^2} - \dfrac{5}{s}$.
37. $\dfrac{3}{s^2 + 2s}$.
38. $\dfrac{13}{s^2 - 9}$.
39. $\dfrac{s^2 + 2s + 5}{(s-1)(s-2)(s-3)}$.
40. $\dfrac{s}{(s+1)(s+2)(s-3)}$.

Example 8.16 Find $\mathscr{L}^{-1}[1/\{s(s^2+9)\}]$.

Solution Let $F(s) = 1/(s^2+9)$. Then $f(t) = \mathscr{L}^{-1}[F(s)] = (\sin 3t)/3$. Therefore, using Eq. (8.20) we obtain

$$\mathscr{L}^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t \frac{1}{3}\sin 3\tau\, d\tau = -\left[\frac{\cos 3\tau}{9}\right]_0^t = \frac{1}{9}(1-\cos 3t).$$

Example 8.17 Find $\mathscr{L}^{-1}\left[\dfrac{1}{s^2(s^2+4)}\right]$

Solution Let $F(s) = 1/(s^2+4)$. Then $f(t) = \mathscr{L}^{-1}[F(s)] = (\sin 2t)/2$. Therefore, using Eq. (8.20) we obtain

$$\mathscr{L}^{-1}\left[\frac{1}{s}F(s)\right] = \mathscr{L}^{-1}\left[\frac{1}{s(s^2+4)}\right] = \int_0^t \frac{1}{2}\sin 2\tau\, d\tau = -\frac{1}{4}[\cos 2\tau]_0^t$$

$$= \frac{1}{4}[1-\cos 2t]$$

and

$$\mathscr{L}^{-1}\left[\frac{1}{s^2}F(s)\right] = \mathscr{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \int_0^t \frac{1}{4}(1-\cos 2\tau)\, d\tau$$

$$= \frac{1}{4}\left[\tau - \frac{\sin 2\tau}{2}\right]_0^t = \frac{1}{8}[2t - \sin 2t].$$

Alternately, we can write

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right]$$

and

$$\mathscr{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \frac{1}{4}\left[\mathscr{L}^{-1}\left(\frac{1}{s^2}\right) - \mathscr{L}^{-1}\left(\frac{1}{s^2+4}\right)\right] = \frac{1}{4}\left[t - \frac{1}{2}\sin 2t\right].$$

Example 8.22 Find the inverse Laplace transforms of the following functions.

(a) $\dfrac{3s-1}{(s-2)^2}$, (b) $\dfrac{6+s}{s^2+6s+13}$.

Solution

(a) We write $\dfrac{3s-1}{(s-2)^2} = \dfrac{3(s-2)+5}{(s-2)^2} = \dfrac{3}{(s-2)} + \dfrac{5}{(s-2)^2}$.

Since $\mathcal{L}[t] = 1/s^2$, we obtain

$$\mathcal{L}^{-1}\left[\frac{3}{s-2} + \frac{5}{(s-2)^2}\right] = \mathcal{L}^{-1}\left[\frac{3}{s-2}\right] + 5\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right] = 3e^{2t} + 5te^{2t}.$$

(b) We write $\dfrac{6+s}{s^2+6s+13} = \dfrac{6+s}{(s+3)^2+4} = \dfrac{(s+3)+3}{(s+3)^2+4}$

$$= \frac{s+3}{(s+3)^2+2^2} + \frac{3}{(s+3)^2+2^2}.$$

Therefore, $\mathcal{L}^{-1}\left[\dfrac{s+3}{(s+3)^2+2^2} + \dfrac{3}{(s+3)^2+2^2}\right] = e^{-3t}\cos 2t + \dfrac{3}{2}e^{-3t}\sin 2t.$

Exercise 8.2

Using the formulas of Laplace transforms of derivatives find the Laplace transforms of the following functions.

1. $\cos^2 2t$.
2. te^{at}
3. $t \sin at$.
4. $t \cos at$.
5. $t \sinh at$.

Use Theorem 8.3 to find the following Laplace transforms.

6. Given that $\mathcal{L}[\cos at] = s/(s^2 + a^2)$, find $\mathcal{L}[\sin at]$.
7. Given that $\mathcal{L}[\sin at] = a/(s^2 + a^2)$, find $\mathcal{L}[\cos at]$.

Using the result in Remark 2, find the following inverse Laplace transforms.

8. Given that $\mathcal{L}^{-1}[1/(s^2 + a^2)] = (\sin at)/a$, find $\mathcal{L}^{-1}[s/(s^2 + a^2)]$.
9. Given that $\mathcal{L}^{-1}[1/\{(s + 1)(s + 2)\}] = e^{-t} - e^{-2t}$, find $\mathcal{L}^{-1}[s/\{(s + 1)(s + 2)\}]$.
10. Find $\mathcal{L}^{-1}[1/(4s^2 + 1)]$. Hence, find $\mathcal{L}^{-1}[2s/(4s^2 + 1)]$.
11. Find $\mathcal{L}^{-1}[1/\{(s^2 + 1)(s^2 + 4)\}]$. Hence, find $\mathcal{L}^{-1}[s/\{(s^2 + 1)(s^2 + 4)\}]$.
12. Find $\mathcal{L}^{-1}[1/\{(s - 1)(s^2 + 1)\}]$. Hence, find $\mathcal{L}^{-1}[s/\{(s - 1)(s^2 + 1)\}]$.

Using Theorem 8.5, find the inverse Laplace transform of the following functions.

13. $\dfrac{1}{s^2 + 5s}$.
14. $\dfrac{16}{s^3 + 9s}$.
15. $\dfrac{s - 2}{s(s + 3)}$.
16. $\dfrac{1}{s^3 + s^2}$.
17. $\dfrac{\omega}{s^2(s^2 + \omega^2)}$.
18. $\dfrac{s - a}{s^2(s^2 + a^2)}$.
19. $\dfrac{3}{s^2(s^2 + 4)(s^2 + 1)}$.
20. $\dfrac{1}{s^4 + 3s^3}$.

Exercise 8.3

Find the Laplace transform of the following functions.

1. $(t^2 - 2t - 3)e^{2t}$.
2. $t^5 e^{-4t}$.
3. $(t - 2)^2 e^{3t}$.
4. $e^t \sin 5t$.
5. $e^{-3t} \cos 3t$.
6. $e^{-t}(\cos t - \sin t)$.
7. $t^3 \cosh 3t$
8. $\sinh t \cos t$.
9. $t^2 \sinh t$.
10. $\cosh \omega t \cos \omega t$.
11. $\sinh t \sin t$.
12. $\cosh \omega t \sin \omega t$.

Find the inverse Laplace transform of the following functions.

13. $\dfrac{1}{s^2 + 6s + 15}$.
14. $\dfrac{1}{s^2 - 4s + 20}$.
15. $\dfrac{s}{s^2 + 4s + 8}$.
16. $\dfrac{5s + 6}{(s - 1)^2}$.
17. $\dfrac{s}{(s - 2)^3}$.
18. $\dfrac{16 + 3s}{s^2 - 8s + 20}$.
19. $\dfrac{3s + 2}{(s + 3)^3}$.
20. $\dfrac{(s + 1)^2}{(s - 2)^4}$.
21. $\dfrac{3s + 5}{s^2 + 6s + 12}$.

Example 8.32 Find the Laplace transforms of the following functions.

(a) $t \sin 4t$, (b) $t^2 \cos 3t$, (c) te^{5t}, (d) $t^2 e^{-2t}$.

Solution

(a) Since $\mathcal{L}(\sin 4t) = 4/(s^2 + 16)$, we obtain

$$\mathcal{L}[t \sin 4t] = -\frac{d}{ds}\left[\frac{4}{s^2 + 16}\right] = \frac{8s}{(s^2 + 16)^2}.$$

(b) Since $\mathcal{L}(\cos 3t) = s/(s^2 + 9)$, we obtain

$$\mathcal{L}[t \cos 3t] = -\frac{d}{ds}\left[\frac{s}{s^2 + 9}\right] = \frac{s^2 - 9}{(s^2 + 9)^2}$$

$$\mathcal{L}[t^2 \cos 3t] = -\frac{d}{ds}\left[\frac{s^2 - 9}{(s^2 + 9)^2}\right] = -\frac{(s^2 + 9)^2(2s) - (s^2 - 9)2(s^2 + 9)(2s)}{(s^2 + 9)^4}$$

$$= -\frac{2s(s^2 + 9) - 4s(s^2 - 9)}{(s^2 + 9)^3} = \frac{2s(s^2 - 27)}{(s^2 + 9)^3}.$$

(c) We have $\mathcal{L}[e^{5t}] = 1/(s - 5)$. Hence

$$\mathcal{L}[te^{5t}] = -\frac{d}{ds}\left[\frac{1}{s - 5}\right] = \frac{1}{(s - 5)^2}.$$

(d) We have $\mathcal{L}[e^{-2t}] = 1/(s + 2)$. Hence

$$\mathcal{L}[t^2 e^{-2t}] = (-1)^2 \frac{d^2}{ds^2}\left[\frac{1}{s + 2}\right] = \frac{2}{(s + 2)^3}.$$

Example 8.33 Find the inverse Laplace transform of the following functions.

(a) $\dfrac{2(s + 1)}{(s^2 + 2s + 2)^2}$, (b) $\dfrac{1}{(s + 5)^4}$. (c) $\dfrac{1}{(s^2 + 9)^2}$.

Solution

(a) We have $\dfrac{2(s + 1)}{(s^2 + 2s + 2)^2} = \dfrac{2(s + 1)}{[(s + 1)^2 + 1]^2} = -\dfrac{d}{ds}\left[\dfrac{1}{(s + 1)^2 + 1}\right] = -F'(s)$.

Hence, $\mathcal{L}^{-1}[-F'(s)] = tf(t)$ and

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{(s + 1)^2 + 1}\right] = e^{-t} \sin t.$$

Therefore, $\mathcal{L}^{-1}\left[\dfrac{2(s + 1)}{(s^2 + 2s + 2)^2}\right] = te^{-t} \sin t.$

(b) We have $\dfrac{1}{(s + 5)^4} = -\dfrac{1}{6}\dfrac{d^3}{ds^3}\left[\dfrac{1}{s + 5}\right]$

and $$\mathcal{L}^{-1}\left[\frac{d^3}{ds^3}\left(\frac{1}{s + 5}\right)\right] = (-1)^3 t^3 e^{-5t}$$ (using Eq. 8.

Therefore, $\mathcal{L}^{-1}\left[\dfrac{1}{(s + 5)^4}\right] = \dfrac{1}{6}t^3 e^{-5t}.$

(c) We write $\dfrac{1}{(s^2 + 9)^2} = \dfrac{(s^2 + 9) + (9 - s^2)}{18(s^2 + 9)^2} = \dfrac{1}{18}\left[\dfrac{1}{s^2 + 9} - \dfrac{s^2 - 9}{(s^2 + 9)^2}\right]$

Therefore, $\mathcal{L}^{-1}\left[\dfrac{1}{(s^2 + 9)^2}\right] = \dfrac{1}{18}\left[\mathcal{L}^{-1}\left(\dfrac{1}{s^2 + 9}\right) - \mathcal{L}^{-1}\left(\dfrac{s^2 - 9}{(s^2 + 9)^2}\right)\right]$

$$= \frac{1}{18}\left[\frac{1}{3}\sin 3t - t \cos 3t\right].$$

(see Example 8.32 (b)).

Example 8.37 Find the inverse Laplace transform of the following functions.

(a) $\ln \frac{s+c}{s+d}$; c, d constants, (b) $\frac{1}{s(s+3)^2}$, (c) $\frac{s}{(s^2-9)^2}$.

Solution

(a) Denote $G(s) = \ln [(s+c)/(s+d)]$. Let $F(s) = -d[G(s)]/ds$.

Therefore,
$$F(s) = -\frac{d}{ds}[\ln (s+c) - \ln (s+d)] = \frac{1}{s+d} - \frac{1}{s+c}.$$

We note that

$$\int_s^{\infty} F(s^*)ds^* = -\int_s^{\infty} \frac{d}{ds} G(s^*)\, ds^* = \int_s^{\infty} \left[\frac{1}{s^*+d} - \frac{1}{s^*+c}\right] ds^* = \left[\ln \frac{s^*+d}{s^*+c}\right]_s^{\infty} = \ln \frac{s+c}{s+d}.$$

Now, $f(t) = \mathcal{L}^{-1}[F(s)] = e^{-dt} - e^{-ct}$. Therefore,

$$\mathcal{L}^{-1}\left[\ln\left(\frac{s+c}{s+d}\right)\right] = \mathcal{L}^{-1}\left[\int_s^{\infty} F(s^*)ds^*\right] = \frac{f(t)}{t} = \frac{1}{t}(e^{-dt} - e^{-ct}).$$

(b) Let $G(s) = \frac{1}{s(s+3)^2}$ and $F(s) = -\frac{d}{ds}G(s) = -\frac{d}{ds}\left[\frac{1}{9}\left(\frac{1}{s} - \frac{1}{s+3} - \frac{3}{(s+3)^2}\right)\right]$

$$= \frac{1}{9}\left[\frac{1}{s^2} - \frac{1}{(s+3)^2} - \frac{6}{(s+3)^3}\right].$$

Now,
$$\int_s^{\infty} F(s^*)ds^* = -\int_s^{\infty}\left[\frac{d}{ds}G(s^*)\right]ds^* = -\left[\frac{1}{s^*(s^*+3)^2}\right]_s^{\infty} = \frac{1}{s(s+3)^2}.$$

We have $f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left\{\frac{1}{9}\left[\frac{1}{s^2} - \frac{1}{(s+3)^2} - \frac{6}{(s+3)^3}\right]\right\} = \frac{1}{9}[t - te^{-3t} - 3t^2e^{-3t}]$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+3)^2}\right] = \frac{1}{9t}(t - te^{-3t} - 3t^2e^{-3t}) = \frac{1}{9}(1 - e^{-3t} - 3te^{-3t}).$$

This result could also have been obtained directly.

(c) We shall use Remark 7 to find the inverse. We have

$$f(t) = t\mathcal{L}^{-1}\left[\int_s^{\infty} \frac{s^*}{(s^{*2}-9)^2}\, ds^*\right] = t\mathcal{L}^{-1}\left[\left\{\frac{-1}{2(s^{*2}-9)}\right\}_s^{\infty}\right] = \frac{t}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2-9}\right]$$

$$= \frac{t}{12}\mathcal{L}^{-1}\left[\frac{1}{s-3} - \frac{1}{s+3}\right] = \frac{t}{12}[e^{3t} - e^{-3t}] = \frac{1}{6}t \sinh 3t.$$

Find the Laplace transform of the following functions.

10. $t \sin 2t$.

11. $t^2 e^{3t}$.

12. $t^2 \sinh 3t$.

13. $t e^{-4t} \sin 3t$.

14. $t \int_0^t e^{-2\tau} \cos 3\tau d\tau$.

15. $t \int_0^t e^{-\tau} \sin 2\tau d\tau$.

16. $t^2 \int_0^t e^{-5\tau} d\tau$.

Find the inverse Laplace transform of the following functions.

17. $\dfrac{1}{(s+a)^2}$.

18. $\dfrac{1}{(s-a)^3}$.

19. $\dfrac{8(s+2)}{(s^2+4s+8)^2}$.

20. $\dfrac{1}{(s^2+16)^2}$.

21. $\dfrac{6s}{(s^2-16)^2}$.

22. $\dfrac{6s+5}{(s^2+2s+2)^2}$.

Find the solution of the following differential equations/initial value problems using Laplace transforms. In problems 29 to 31, $y'(0)$ is arbitrary.

23. $ty' - 2y = 6$.

24. $ty' - 3y = 2t$.

25. $y'' - ty' + 4y = 3, \quad y(0) = 0, \quad y'(0) = 0$.

26. $y'' + 4ty' - 12y = 0, \quad y(0), = 0, \quad y'(0) = -2$.

27. $y'' + 6ty' - 12y = 1, \quad y(0) = 2, \quad y'(0) = 0$.

28. $y'' + 6ty' - 30y = 0, \quad y(0) = 0, \quad y'(0) = 2$.

29. $ty'' + 4ty' + 4y = 8, \quad y(0) = 2$.

30. $ty'' + (6t-2)y' - 6y = 0, \quad y(0) = 1$.

31. $ty'' + (8t-2)y' - 8y = 0, \quad y(0) = 2$.

Find the Laplace transform of the following functions.

32. $(\sinh t)/t$.

33. $(1 - \cos bt)/t$.

34. $(e^{-2t} \sin 3t)/t$.

Find the inverse Laplace transform of the following functions.

35. $\dfrac{s}{(s+4)^3}$.

36. $\dfrac{18}{(s^2+9)^2}$.

37. $\ln\left(\dfrac{s^2+1}{s^2}\right)$.

38. $\ln\left(\dfrac{s-1}{s+1}\right)$.

39. $\ln\left(\dfrac{s^2+1}{s(s+1)}\right)$.

40. $\cot^{-1} s$.

41. $\dfrac{1}{s}\tan^{-1}\left(\dfrac{1}{s}\right)$.

42. $\tanh^{-1}\left(\dfrac{1}{s}\right)$.

43. Give an example to show th...

Write the following functions, whose graphs are given, in terms of unit step functions and find their Laplace transforms.

29. Rectangular pulse (Fig. 8.9).

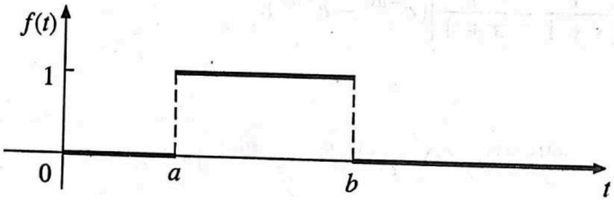

Fig. 8.9. Problem 29.

30. Undamped single square pulse (Fig. 8.10).

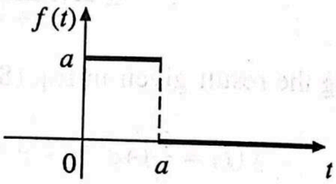

Fig. 8.10. Problem 30.

31.

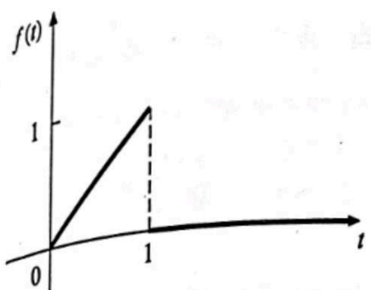

Fig. 8.11. Problem 31.

32. Periodic function of period $2a$.

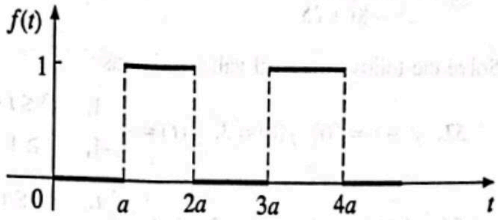

Fig. 8.12. Problem 32.

33. Staircase function.

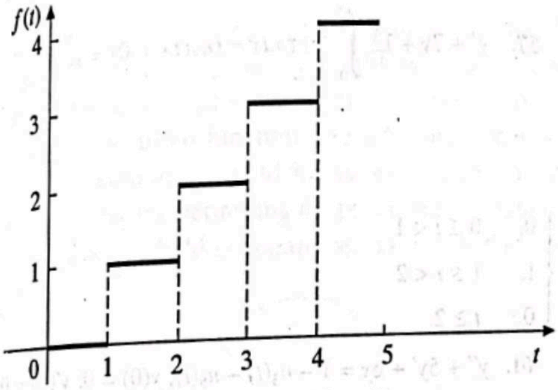

Fig. 8.13. Problem 33.

34.

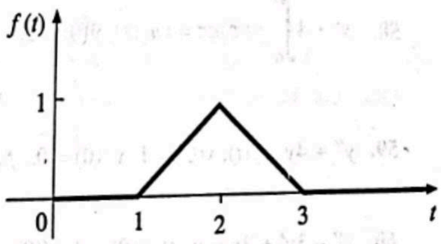

Fig. 8.14. Problem 34.

In each of the following problems, find the Laplace transform of the given functions.

35. $(t^2 - 1)\, u_2(t)$.

36. $(t^2 + 1)u_1(t)$.

37. $(\cos t)\, u_1(t)$.

38. $(\sin t)\, u_\pi(t)$.

39. $\cos(t - 3)\, u_3(t)$.

40. $e^{3-t}u_3(t)$.

41. $f(t) = \begin{cases} 5, & 0 \le t < 2 \\ -5, & t \ge 2. \end{cases}$

42. $f(t) = \begin{cases} t, & 0 \le t < 3 \\ 0, & t \ge 3. \end{cases}$

43. $f(t) = \begin{cases} 0, & 0 \le t < \pi/2 \\ \sin t, & t \ge \pi/2. \end{cases}$

44. $f(t) = \begin{cases} k, & 0 \le t < 2 \\ 0, & 2 \le t < 4 \\ k, & t \ge 4. \end{cases}$

45. $f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ 0, & t \ge 1. \end{cases}$

46. $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t - 1, & 1 \le t < 2 \\ 0, & t \ge 2. \end{cases}$

In each of the following problems, find the inverse Laplace transform of the given functions.

47. $\dfrac{e^{-s}}{s^3}$.

48. $\dfrac{e^{-s\pi/2}}{s^2 + 1}$.

49. $\dfrac{se^{-s\pi}}{s^2+9}$.

50. $\dfrac{(1+e^{-s\pi})^2}{s+5}$.

51. $\dfrac{e^{-s\pi}-e^{-2s\pi}}{s^2-8s+25}$.

52. $\dfrac{s(1+e^{-s\pi/2})}{s^2+4}$.

Find the Laplace transforms of the following functions which are defined graphically.

2. Square wave with period $2a$, as given in Fig. 8.26.

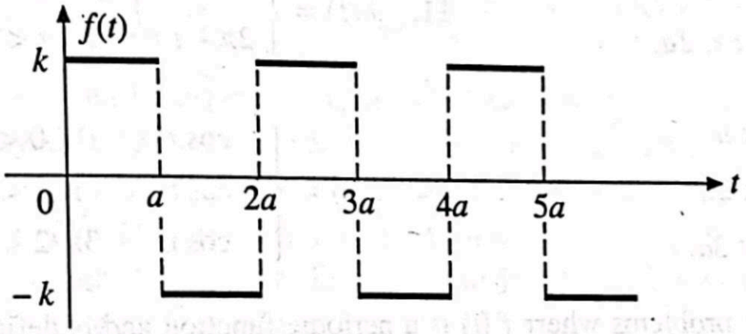

Fig. 8.26. Problem 2.

3. Square wave with period $2a$, as given in Fig. 8.27.

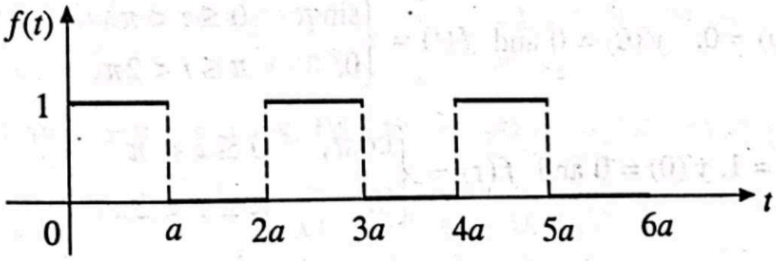

Fig. 8.27. Problem 3.

Find the Laplace transforms of the periodic functions which are defined in one period as the followi

6. $f(t) = kt, \quad 0 < t < 2\pi, \quad k$ a constant.

7. $f(t) = t^2, \quad 0 < t < 2\pi.$

8. $f(t) = \begin{cases} t, & 0 < t < a \\ 0, & a < t < 2a. \end{cases}$

9. $f(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \\ -1, & 2\pi < t < 3\pi \\ 0, & 3\pi < t < 4\pi. \end{cases}$

Find the following convolutions.

46. $1 * e^{-2t}$.

47. $t * e^{at}$.

48. $e^{at} * e^{bt}$ $(a \neq b)$.

49. $\sin \omega t * \sin \omega t$.

In the following problems use the convolution theorem to find the inverse Laplace transform.

50. $\dfrac{1}{(s-a)(s-b)}$.

51. $\dfrac{1}{s^2(s^2+16)}$.

52. $\dfrac{1}{(s^2+9)^2}$.

53. $\dfrac{s}{(s^2+4)^2}$.

54. $\dfrac{s}{(s^2+4)(s^2+9)}$.

55. $\dfrac{6}{(s+1)^2(s+2)}$.

22. Assume that the function $f(t)$ satisfies the conditions of Theorem 8.5. Then, show that

$$\mathscr{L}\left[\int_a^t f(\tau)d\tau\right] = \frac{1}{s}F(s) - \frac{1}{s}\int_0^a f(\tau)d\tau, \quad a > 0, \text{ where } \mathscr{L}[f(t)] = F(s).$$

Solve the following initial value problems.

23. $y' + 3y = 1,\ y(0) = 1.$

24. $y' - 4y = t,\ y(0) = -1.$

25. $y' + 2y = \sin t,\ y(0) = 1.$

26. $y' + 3y = \cos t,\ y(0) = 0.$

27. $y' - 2y = 1 + t,\ y(0) = 2.$

28. $y'' + y = t,\ y(0) = 1,\ y'(0) = 0.$

29. $y'' + 3y' + 2y = 3,\ y(0) = 1,\ y'(0) = 1.$

30. $y'' + 5y' + 4y = e^{3t},\ y(0) = 0,\ y'(0) = 3.$

31. $y'' - 6y' + 5y = e^{2t},\ y(0) = 1,\ y'(0) = -1.$

32. $4y'' - 8y' + 3y = \sin t,\ y(0) = 0,\ y'(0) = 2.$

33. $2y'' - y' - y = \cos t,\ y(0) = 1,\ y'(0) = 0.$

34. $y' - 4y + 3\int_0^t y(\tau)d\tau = t,\ y(0) = 1.$

35. $y' + 6y + 5\int_0^t y(\tau)d\tau = 1 + t,\ y(0) = 1.$

36. $y' - y - 6\int_0^t y(\tau)d\tau = \sin t,\ y(0) = 2.$

Use the Laplace transforms to solve the following initial value problems.

22. $y' + y = 1 + te^t$, $y(0) = 1$.

23. $y'' + 6y' + 9y = 8te^{2t}$, $y(0) = 0$, $y'(0) = -1$.

24. $y'' + 8y' + 16y = te^{-4t}$, $y(0) = 1$, $y'(0) = 2$.

25. $y'' + 6y' + 13y = e^{-t}$, $y(0) = 0$, $y'(0) = 4$.

26. $y'' + y = e^t \sin t$, $y(0) = 0$, $y'(0) = 0$.

27. $y'' + 2y' + 5y = 1 + t$, $y(0) = 4$, $y'(0) = -3$.

28. $12y'' - 24y' + 9y = 2t$, $y(0) = 0$, $y'(0) = 3$.

Find the solution of the following differential equations/initial value problems using Laplace transforms. In problems **29** to **31**, $y'(0)$ is arbitrary.

23. $ty' - 2y = 6$.

24. $ty' - 3y = 2t$.

25. $y'' - ty' + 4y = 3$, $y(0) = 0$, $y'(0) = 0$.

26. $y'' + 4ty' - 12y = 0$, $y(0), = 0$, $y'(0) = -2$.

27. $y'' + 6ty' - 12y = 1$, $y(0) = 2$, $y'(0) = 0$.

28. $y'' + 6ty' - 30y = 0$, $y(0) = 0$, $y'(0) = 2$.

29. $ty'' + 4ty' + 4y = 8$, $y(0) = 2$.

30. $ty'' + (6t - 2)y' - 6y = 0$, $y(0) = 1$.

31. $ty'' + (8t - 2)y' - 8y = 0$, $y(0) = 2$.

Solve the following initial value problems.

53. $y' + y = f(t),\ y(0) = 3,\ f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & t \ge 1. \end{cases}$

54. $y' + 3y = f(t),\ y(0) = 2,\ f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1. \end{cases}$

55. $y' + 2y = f(t),\ y(0) = 1,\ f(t) = \begin{cases} 1, & 0 \le t < 2 \\ 0, & t \ge 2. \end{cases}$

56. $y' + 4y + 4\int_0^t y(\tau)d\tau = u_1(t),\ y(0) = 3.$

57. $y' + 7y + 12\int_0^t y(\tau)d\tau = tu_2(t),\ y(0) = 1.$

58. $y' + 4\int_0^t y(\tau)d\tau = t\,u_1(t),\ y(0) = 2.$

59. $y'' + 4y = f(t),\ y(0) = 1,\ y'(0) = 0,\ f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 0, & t \ge 2. \end{cases}$

60. $y'' - 3y' + 2y = u_1(t),\ y(0) = 1,\ y'(0) = 1.$

61. $y'' + 5y' + 6y = 1 - u_3(t) - u_5(t),\ y(0) = 0,\ y'(0) = 0.$

62. $y'' + 4y' + 3y = f(t),\ y(0) = 0,\ y'(0) = 1,\ f(t) = \begin{cases} -1, & 0 \le t < 3 \\ 0, & t \ge 3. \end{cases}$

63. The integro-differential equation governing the flow of current $i(t)$ in an RC - circuit (Fig. 8.15) is given by (where the resistance R and capacitance C are constants)

$$Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t).$$

If initially at $t = 0$ there is no current and $E(t) = v[u_1(t) - u_2(t)]$, v constant, find the current $i(t)$.

64. The differential equation governing the flow of current $i(t)$ in an LR series circuit (Fig. 8.16) is given by (where the inductance L and resistance R are constants)

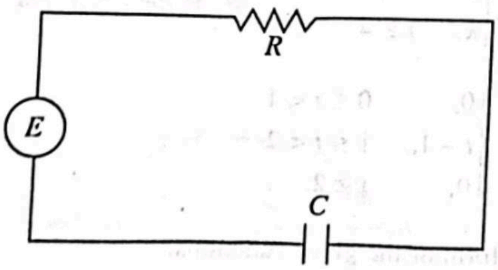

Fig. 8.15. RC series circuit.

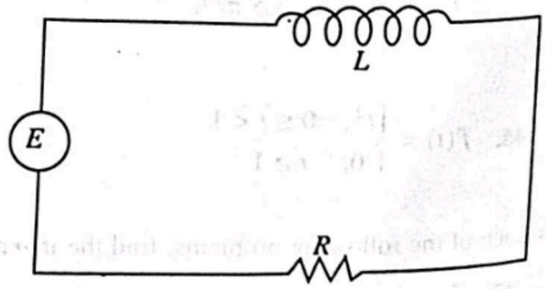

Fig. 8.16. LR series circuit.

$$L\frac{di}{dt} + Ri = E(t).$$

Find the current $i(t)$ if the current is initially zero and $E(t) = \begin{cases} 0, & 0 \le t < 2, \\ 5, & t \ge 2. \end{cases}$

65. Solve for the current in the circuit of problem 64, if the current is initially zero and

$$E(t) = \begin{cases} \sin t, & 0 \le t < \pi/2, \\ 0, & t \ge \pi/2. \end{cases}$$

Solve the following initial value problems where $f(t)$ is a periodic function and is defined in one period.

14. $y'' + 4y' + 5y = f(t), \quad y(0) = 0, \quad y'(0) = 0$, and $f(t) = \begin{cases} 1, & 0 \le t < \pi \\ -1, & \pi \le t < 2\pi. \end{cases}$

15. $y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 0$ and $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi. \end{cases}$

16. $y'' + 9y = f(t), y(0) = 1, y'(0) = 0$ and $f(t) = \begin{cases} \cos t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi. \end{cases}$

17. $y'' + 6y' + 10y = f(t), \quad y(0) = 0, \quad y'(0) = 1$ and $f(t) = \begin{cases} 1, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi. \end{cases}$

18. $y'' + 3y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 1$ and $f(t) = t, \quad 0 \le t < a.$

19. $y'' + 4y' + 3y = f(t), \quad y(0) = 0, \quad y'(0) = 0$ and $f(t) = \begin{cases} t, & 0 < t \le \pi \\ 2\pi - t, & \pi < t < 2\pi. \end{cases}$

20. $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0$ and $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ \sin t, & \pi \le t < 2\pi. \end{cases}$

Exercise 8.2

1. $f(t) = \cos^2 2t$, $f(0) = 1$, $f'(t) = -2\sin 4t$; $F(s) = (s^2 + 8)/[s(s^2 + 16)]$.

2. $f(t) = te^{at}$, $f(0) = 0$, $f'(t) = af(t) + e^{at}$; $F(s) = 1/(s - a)^2$.

3. $f(t) = t\sin at$, $f(0) = 0$, $f'(0) = 0$, $f'' = -a^2 f(t) + 2a\cos at$; $F(s) = 2as/(s^2 + a^2)^2$.

4. $f(t) = t\cos at$, $f(0) = 0$, $f'(0) = 1$, $f'' = -a^2 f(t) - 2a\sin at$; $F(s) = (s^2 - a^2)/(s^2 + a^2)^2$.

5. $f(t) = t\sinh at$, $f(0) = 0$, $f'(0) = 0$, $f'' = a^2 f(t) + 2a\cosh at$; $F(s) = 2as/(s^2 - a^2)^2$.

6. $f(t) = \cos at$, $f(0) = 1$, $f' = -a\sin at$.

7. $f(t) = \sin at$, $f(0) = 0$, $f' = a\cos at$.

8. $f'(t) = [(\sin at)/a]' = \cos at$.

9. $f'(t) = (e^{-t} - e^{-2t})' = 2e^{-2t} - e^{-t}$.

10. $\mathcal{L}^{-1}[1/(4s^2 + 1)] = [\sin(t/2)]/2 = f(t)$; $f'(t) = [\cos(t/2)]/4$.

11. $\mathcal{L}^{-1}\left[\dfrac{1}{(s^2+1)(s^2+4)}\right] = \dfrac{1}{3}\left[\sin t - \dfrac{1}{2}\sin 2t\right] = f(t)$; $f'(t) = \dfrac{1}{3}[\cos t - \cos 2t]$.

12. $\mathcal{L}^{-1}\left[\dfrac{1}{(s-1)(s^2+1)}\right] = \dfrac{1}{2}[e^t - \cos t - \sin t]$; $f'(t) = \dfrac{1}{2}[e^t + \sin t - \cos t]$.

13. Write as $F(s)/s$ where $F(s) = 1/(s+5)$; $(1 - e^{-5t})/5$.

14. Write as $F(s)/s$ where $F(s) = 16/(s^2 + 9)$; $16(1 - \cos 3t)/9$.

15. Write as $\dfrac{1}{s} - \dfrac{5}{s(s+1)} = \dfrac{1}{s} - \dfrac{F(s)}{s}$ where $F(s) = \dfrac{5}{s+1}$; $-\dfrac{1}{3}(2 - 5e^{-3t})$.

16. Write as $\dfrac{1}{s}\left[\dfrac{1}{s}F(s)\right]$ where $F(s) = 1/(s+1)$; $t + e^{-t} - 1$.

17. Write as $\dfrac{1}{s}\left[\dfrac{1}{s}F(s)\right]$ where $F(s) = \dfrac{\omega}{s^2 + \omega^2}$; $\dfrac{1}{\omega^2}(\omega t - \sin \omega t)$.

18. Write as $\dfrac{1}{s}\left[\dfrac{1}{s}\left\{\dfrac{s}{s^2+a^2} - \dfrac{a}{s^2+a^2}\right\}\right]$; $\dfrac{1}{a^2}[\sin at - \cos at + 1] - \dfrac{t}{a}$.

19. Write as $\dfrac{1}{s}\left[\dfrac{1}{s}\left\{\dfrac{1}{s^2+1} - \dfrac{1}{s^2+4}\right\}\right]$; $\dfrac{3}{4}t - \sin t + \dfrac{1}{8}\sin 2t$.

20. Write as $\dfrac{1}{s}\left[\dfrac{1}{s}\left\{\dfrac{1}{s}\cdot\dfrac{1}{s+3}\right\}\right] = \dfrac{1}{18}(3t^2 - 2t) + \dfrac{1}{27}(1 - e^{-3t})$.

21. Deduce from the discussion after Eq. (8.8).

22. Write $\displaystyle\int_a^t f(\tau)d\tau = \int_0^t f(\tau)d\tau - \int_0^a f(\tau)d\tau$. The second integral is a constant.

23. $Y(s) = \dfrac{1}{3s} + \dfrac{2}{3(s+3)}$, $y(t) = \dfrac{1}{3} + \dfrac{2}{3}e^{-3t}$.

24. $Y(s) = -\dfrac{1}{16s} - \dfrac{1}{4s^2} - \dfrac{15}{16(s-4)}$, $y(t) = -\dfrac{1}{16} - \dfrac{1}{4}t - \dfrac{15}{16}e^{4t}$.

25. $Y(s) = -\dfrac{s}{5(s^2+1)} + \dfrac{2}{5(s^2+1)} + \dfrac{6}{5(s+2)}$, $y(t) = -\dfrac{1}{5}\cos t + \dfrac{2}{5}\sin t + \dfrac{6}{5}e^{-2t}$.

26. $Y(s) = -\dfrac{3}{10(s+3)} + \dfrac{3s}{10(s^2+1)} + \dfrac{1}{10(s^2+1)}$, $y(t) = -\dfrac{3}{10}e^{-3t} + \dfrac{3}{10}\cos t + \dfrac{1}{10}\sin t$.

27. $Y(s) = \dfrac{11}{4(s-2)} - \dfrac{3}{4s} - \dfrac{1}{2s^2}$, $y(t) = \dfrac{11}{4}e^{2t} - \dfrac{3}{4} - \dfrac{t}{2}$.

28. $Y(s) = \dfrac{s}{s^2+1} + \dfrac{1}{s^2} - \dfrac{1}{s^2+1}$, $y(t) = \cos t + t - \sin t$.

29. $Y(s) = \dfrac{3}{2s} - \dfrac{1}{2(s+2)}$, $y(t) = \dfrac{3}{2} - \dfrac{1}{2}e^{-2t}$.

30. $Y(s) = \dfrac{11}{12(s+1)} - \dfrac{20}{21(s+4)} + \dfrac{1}{28(s-3)}$, $y(t) = \dfrac{11}{12}e^{-t} - \dfrac{20}{21}e^{-4t} + \dfrac{1}{28}e^{3t}$.

31. $Y(s) = -\dfrac{1}{3(s-2)} - \dfrac{5}{12(s-5)} + \dfrac{7}{4(s-1)}$, $y(t) = \dfrac{7}{4}e^{t} - \dfrac{5}{12}e^{5t} - \dfrac{1}{3}e^{2t}$.

32. $Y(s) = \dfrac{8s}{65(s^2+1)} - \dfrac{1}{65(s^2+1)} - \dfrac{11}{5(s-1/2)} + \dfrac{27}{13(s-3/2)}$

$y(t) = \dfrac{8}{65}\cos t - \dfrac{1}{65}\sin t - \dfrac{11}{5}e^{t/2} + \dfrac{27}{13}e^{3t/2}$.

33. $Y(s) = -\dfrac{3s}{10(s^2+1)} - \dfrac{1}{10(s^2+1)} + \dfrac{1}{2(s-1)} + \dfrac{8}{5(2s+1)}$,

$y(t) = -\dfrac{3}{10}\cos t - \dfrac{1}{10}\sin t + \dfrac{1}{2}e^{t} + \dfrac{4}{5}e^{-t/2}$.

34. $Y(s) = \dfrac{1}{3s} + \dfrac{5}{3(s-3)} - \dfrac{1}{s-1}$, $y(t) = \dfrac{1}{3} - e^{t} + \dfrac{5}{3}e^{3t}$.

35. $Y(s) = \dfrac{1}{5s} - \dfrac{1}{4(s+1)} + \dfrac{21}{20(s+5)}$, $y(t) = \dfrac{1}{5} - \dfrac{1}{4}e^{-t} + \dfrac{21}{20}e^{-5t}$.

36. $Y(s) = -\dfrac{7s}{50(s^2+1)} - \dfrac{1}{50(s^2+1)} + \dfrac{63}{50(s-3)} + \dfrac{22}{25(s+2)}$,

$y(t) = -\dfrac{7}{50}\cos t - \dfrac{1}{50}\sin t + \dfrac{63}{50}e^{3t} + \dfrac{22}{25}e^{-2t}$.

37. $Y_1(s) = \dfrac{1}{4s^2} - \dfrac{s}{s^2+4} + \dfrac{7}{4(s^2+4)}$, $y_1(t) = \dfrac{t}{4} - \cos 2t + \dfrac{7}{8}\sin 2t$,

$Y_2(s) = -\dfrac{3}{4s} + \dfrac{7s}{4(s^2+4)} + \dfrac{4}{s^2+4}$, $y_2(t) = -\dfrac{3}{4} + \dfrac{7}{4}\cos 2t + 2\sin 2t$.

38. $Y_1(s) = \dfrac{3}{s^2+9} - \dfrac{3s}{s^2+9} + \dfrac{9}{s(s^2+9)}$, $y_1(t) = 1 + \sin 3t - 4\cos 3t$,

$Y_2(s) = \dfrac{s}{s^2+9} + \dfrac{9}{s^2+9}$, $y_2(t) = \cos 3t + 3\sin 3t$.

39. $Y_2(s) = \dfrac{s}{s^2+9} + \dfrac{9}{s^2+9}$, $y_2(t) = \cos 3t + 3\sin 3t$.

$$Y_2(s) = \frac{2}{s^2+1} - \frac{s}{s^2+1} + \frac{1}{s}, \quad y_2(t) = 2\sin t - \cos t + 1.$$

40. $Y_1(s) = \frac{2}{s^3(1+s^2)} + \frac{2}{s(1+s^2)} - \frac{1}{1+s^2}.$

$y_1(t) = 2(t^2/2 + \cos t - 1) + 2(1 - \cos t) - \sin t = t^2 - \sin t.$

$Y_2(s) = \frac{1}{s^2(1+s^2)} + \frac{s}{1+s^2} + \frac{1}{1+s^2}$, $y_2(t) = (t - \sin t) + \cos t + \sin t = t + \cos t.$

Exercise 8.3

1. $\frac{2}{(s-2)^3} - \frac{2}{(s-2)^2} - \frac{3}{s-2}.$

2. $\frac{120}{(s+4)^6}.$

3. $\frac{2}{(s-3)^3} - \frac{4}{(s-3)^2} + \frac{4}{(s-3)}.$

4. $\frac{5}{s^2 - 2s + 26}.$

5. $\frac{s+3}{s^2+6s+18}.$

6. $\frac{s}{s^2+2s+2}.$

In Problems 7 to 12 write sinh *at*, cosh *at* in terms of exponential functions.

7. $\frac{6(s^4+54s^2+81)}{(s^2-9)^4}.$

8. $\frac{s^2-2}{s^4+4}.$

9. $\frac{2(3s^2+1)}{(s^2-1)^3}.$

10. $\frac{s^3}{s^4+4\omega^4}.$

11. $\frac{2s}{s^4+4}.$

12. $\frac{\omega(s^2+2\omega^2)}{s^4+4\omega^4}.$

13. $(e^{-3t}\sin\sqrt{6}t)/\sqrt{6}.$

14. $(e^{2t}\sin 4t)/4.$

15. $(\cos 2t - \sin 2t)\, e^{-2t}.$

16. $(5 + 11t)e^t.$

17. $(t + t^2)e^{2t}.$

18. $(3\cos 2t + 14\sin 2t)\, e^{4t}.$

19. $(6t - 7t^2)\, e^{-3t}/2.$

20. $(2t + 6t^2 + 3t^3)\, e^{2t}/2.$

21. $(3\sqrt{3}\cos\sqrt{3}t - 4\sin\sqrt{3}t)e^{-3t}/\sqrt{3}.$

22. $1 + [e^{-t} - e^t + 2te^t]/4.$

23. $[(16 - 85t)\, e^{-3t} + (40t - 16)\, e^{2t}]/125.$

24. $e^{-4t}(6 + 36t + t^3)/6.$

25. $[e^{-t} + (15\sin 2t - \cos 2t)\, e^{-3t}]/8.$

26. $[(2\cos t + \sin t) + e^t(\sin t - 2\cos t)]/5.$

27. $[6 + 10t + (17\sin 2t + 194\cos 2t)\, e^{-t}]/50.$

28. $[16 + 6t + \{182\sinh(t/2) - 16\cosh(t/2)\}e^t]/27.$

29. $f(t) = u_a(t) - u_b(t)$, $(e^{-as} - e^{-bs})/s.$

30. $f(t) = a[u_0(t) - u_a(t)]$, $a(1 - e^{-as})/s.$

31. $f(t) = t[u_0(t) - u_1(t)]$, $[1 - e^{-s}(s+1)]/s^2.$

32. $f(t) = u_a(t) - u_{2a}(t) + u_{3a}(t) - u_{4a}(t) + \ldots$, $e^{-as/2}/[2s\cosh(as/2)].$

33. $e^{-s/2}/[(2s)\sinh(s/2)].$

34. $e^{-s}(1 - e^{-s})^2/s^2.$

35. $e^{-2s}(2 + 4s + 3s^2)/s^3.$

36. $2e^{-s}(1 + s + s^2)/s^3.$

37. $e^{-s}(s\cos 1 - \sin 1)/(s^2+1).$

38. $-e^{-\pi s}/(s^2+1).$

39. $se^{-3s}/(s^2+1)$.
40. $e^{-3s}/(s+1)$.
41. $5(1-2e^{-2s})/s$.
42. $[1-e^{-3s}(1+3s)]/s^2$.
43. $se^{-\pi s/2}/(s^2+1)$.
44. $k[1-e^{-2s}+e^{-4s}]/s$.
45. $[2-e^{-s}(2+2s+s^2)]/s^3$.
46. $[e^{-s}-e^{-2s}(1+s)]/s^2$.
47. $u_1(t)\,(t-1)^2/2$.
48. $-\cos t\, u_{\pi/2}(t)$.
49. $-\cos 3t\, u_\pi(t)$.
50. $e^{-5t}[u_0(t)+2e^{5\pi}u_\pi(t)+e^{10\pi}u_{2\pi}(t)]$.
51. $-e^{4t}\sin 3t\,[e^{-4\pi}u_\pi(t)+e^{-8\pi}u_{2\pi}(t)]/3$.
52. $\cos 2t\,[u_0(t)-u_{\pi/2}(t)]$.
53. $(1+2e^{-t})u_0(t)-2[1-e^{-(t-1)}]u_1(t)$.
54. $[(3t-1+19e^{-3t})\,u_0(t)+(1-3t+2e^{-3(t-1)})u_1(t)]/9$.
55. $[(1+e^{-2t})\,u_0(t)-(1-e^{-2(t-2)})\,u_2(t)]/2$.
56. $(3e^{-2t}-6te^{-2t})\,u_0(t)+(t-1)\,e^{-2(t-1)}u_1(t)$.
57. $(4e^{-4t}-3e^{-3t})u_0(t)+u_2(t)\,(1+20\,e^{-3(t-2)}-21\,e^{-4(t-2)})/12$.
58. $2\cos 2t\,u_0(t)+[1-\cos 2(t-1)+2\sin 2\,(t-1)]\,u_1(t)/4$.
59. $u_0(t)\cos 2t+[\{1-\cos 2(t-1)\}\,u_1(t)-\{1-\cos 2(t-2)\}\,u_2(t)]/4$.
60. $e^tu_0(t)+[1-2e^{(t-1)}+e^{2(t-1)}]u_1(t)/2$.
61. $(1/6)\,[1+2e^{-3t}-3e^{-2t}]u_0(t)-(1/6)\,[1+2e^{-3(t-3)}-3e^{-2(t-3)}]\,u_3(t)-(1/6)\,[1+2e^{-3(t-5)}-3e^{-2(t-5)}]\,u_5(t)$.
62. $(1/3)\,(3e^{-t}-2e^{-3t}-1)\,u_0(t)+(1/6)[2+e^{-3(t-3)}-3e^{-(t-3)}]\,u_3(t)$.
63. $(v/R)\,[e^{-p(t-1)}u_1(t)-e^{-p(t-2)}\,u_2(t)],\ \ p=1/(RC)$.
64. $(5/R)\,[1-e^{-p(t-2)}]\,u_2(t),\ \ p=R/L$.
65. $k[e^{-pt}-\cos t+p\sin t]\,u_0(t)-k[p\sin t-\cos t-pe^{-p(t-\pi/2)}]u_{\pi/2}(t)$.
    $p=R/L$ and $k=1/[L(1+p^2)]$.

Exercise 8.4

1. 0.
2. $1/\sqrt{2}$.
3. 2.
4. $e^{-2(t-3)}\sin(t-3)\,u_3(t)$.
5. $[\sin 4\,(t-2)\,u_2(t)]/4$.
6. $2e^{-(t-2)}\sin 3(t-2)\,u_2(t)-e^{-(t-3)}\sin 3(t-3)\,u_3(t)$.
7. $[4\sin 3t]/3$.
8. $8e^{-2(t-1)}\sin 2(t-1)\,u_1(t)+4e^{-2(t-2)}\sin 2(t-2)\,u_2(t)$.
9. $2\cos 2t+4\sin 2\,(t-\pi/6)\,u_{\pi/6}(t)$.
10. $4s/(s^2+4)^2$.
11. $2/(s-3)^3$.
12. $18(s^2+3)/(s^2-9)^3$.
13. $6(s+4)/(s^2+8s+25)^2$.
14. $2(s^3+3s^2+8s+13)/[s^2(s^2+4s+13)^2]$.
15. $2(3s^2+4s+5)/[s^2(s^2+2s+5)^2]$.
16. $2(3s^2+15s+25)/[s^3(s+5)^3]$.
17. te^{-at}.
18. $t^2e^{at}/2$.
19. $2te^{-2t}\sin 2t$.
20. $(\sin 4t-4t\cos 4t)/128$.
21. $(3t\sinh 4t)/4$.

22. $3te^{-t}\sin t - (e^{-t}\sin t - te^{-t}\cos t)/2$. Write the given expression as

$$\frac{6(s+1)}{[(s+1)^2+1]^2} - \frac{1}{2}\left[\frac{1}{(s+1)^2+1} - \frac{(s+1)^2-1}{\{(s+1)^2+1\}^2}\right].$$

23. $-3 + (ct^2/2)$, c arbitrary constant.

24. $-t + (ct^3/6)$, c arbitrary constant.

25. $(6t^2 - t^4)/4$.

26. $-2\,(3t + 4t^3)/3$.

27. $(4 + 25\,t^2)/2$.

28. $2(5t + 20t^3 + 12t^5)/5$.

29. $2 + (16 - c)\,te^{-4t}$, c arbitrary constant.

30. $(1 + 3t)\,e^{-6t} + c\,[-1 + 3t + (1 + 3t)\,e^{-6t}]/108$, c arbitrary constant.

31. $2(1 + 4t)\,e^{-8t} + c\,[-1 + 4t + (1 + 4t)\,e^{-8t}]/256$, c arbitrary constant.

32. $\frac{1}{2}\ln\left(\frac{s+1}{s-1}\right)$.

33. $\ln\left(\frac{\sqrt{s^2+b^2}}{s}\right)$.

34. $\cot^{-1}[(s+2)/3]$.

35. $t(1-2t)e^{-4t}$.

36. $(\sin 3t - 3t\cos 3t)/3$.

37. $2(1 - \cos t)/t$.

38. $-(2\sinh t)/t$.

39. $(1 + e^{-t} - 2\cos t)/t$.

40. $(\sin t)/t$.

41. $\int_0^t \frac{\sin\tau}{\tau}\,d\tau$

42. $(\sinh t)/t$.

43. $f(t) = \begin{cases} \cos t, & (\pi/2) \le t \le \pi \\ 0, & \text{otherwise.} \end{cases}$

46. $(1 - e^{-2t})/2$.

47. $(e^{at} - 1 - at)/a^2$.

48. $(e^{at} - e^{bt})/(a - b)$.

49. $[\sin\omega t - \omega t\cos\omega t]/(2\omega)$.

50. $(e^{at} - e^{bt})/(a - b)$.

51. $(4t - \sin 4t)/64$.

52. $(\sin 3t - 3t\cos 3t)/54$.

53. $(t\sin 2t)/4$.

54. $(\cos 2t - \cos 3t)/5$.

55. $6\,[(t-1)e^{-t} + e^{-2t}]$.

56. $(60\,t\sin t - 8\cos t + 8\cos 4t)/225$.

57. $[\{2 + (a-b)t\}\,e^{-at} - \{2 - (a-b)\,t\}\,e^{-bt}]/(a-b)^3$.

In Problems 58 to 70, use convolution theorem for one or both terms of $Y(s)$.

58. $(t\sin 4t)/8$.

59. $(t-2)e^{-t} + 2e^{-2t}$.

60. $(3t+2)e^{-2t} + (t-2)e^{-t}$.

61. $[2\omega\cosh\omega t + 4\sinh\omega t + t\sinh\omega t]/(2\omega)$.

62. $[(34 - 7t)\,e^{-t} + 15\,e^{6t}]/49$.

63. $[t\sin t + \cos t - 1]\,e^t$.

64. $[t\cos t - (1-t)\sin t]/2$.

65. $[-6 + 7t + 6e^{7t}]/49$.

66. $[3e^t + 2te^t + e^{-t}]/4$.

67. $[2e^t - t - 1]$.

68. $2e^{-t} - e^{-t/2}\,[\cos(\sqrt{3}t/2) + (1/\sqrt{3})\sin(\sqrt{3}t/2)]$.

69. $\cos t + \sin t$.

70. $\frac{1}{8}\left[1 + 2t - \left\{\cosh\left(\frac{\sqrt{17}}{4}t\right) + \frac{1}{\sqrt{17}}\sinh\left(\frac{\sqrt{17}}{4}t\right)\right\}e^{-t/4}\right]$.

Exercise 8.5

1. $T = 2\pi/\omega$.

2. $f(t) = \begin{cases} k, 0 < t < a \\ -k, a < t < 2a \end{cases}$, $T = 2a$, $F(s) = \frac{k}{s}\tanh\left(\frac{as}{2}\right)$.

3. $f(t) = \begin{cases} 1, 0 < t < a \\ 0, a < t < 2a \end{cases}$, $T = 2a$, $F(s) = \frac{1}{s(1 + e^{-as})}$.

4. $f(t) = \begin{cases} \sin t, 0 < t < \pi \\ 0, \pi < t < 2\pi \end{cases}$, $T = 2\pi$, $F(s) = \frac{1}{(s^2+1)(1 - e^{-\pi s})}$.

5. $f(t) = \begin{cases} \sin \omega t, 0 < t < \pi/\omega \\ -\sin \omega t, \pi/\omega < t < 2\pi/\omega \end{cases}$, $T = \frac{2\pi}{\omega}$, $F(s) = \frac{\omega \coth[\pi s/(2\omega)]}{s^2 + \omega^2}$.

6. $\frac{k}{s^2} - \frac{2k\pi e^{-2\pi s}}{s(1 - e^{-2\pi s})}$,

7. $\frac{2}{s^3} - \frac{4\pi(\pi s + 1)e^{-2\pi s}}{s^2(1 - e^{-2\pi s})}$.

8. $\frac{(1 - e^{-as}) - ase^{-as}}{s^2(1 - e^{-2as})}$,

9. $\frac{1 - e^{-\pi s} - e^{-2\pi s} + e^{-3\pi s}}{s(1 - e^{-4\pi s})} = \frac{1 - e^{-\pi s}}{s(1 + e^{-2\pi s})}$.

10. $\frac{e^{-as} - (as + 1)e^{-2as}}{s^2(1 - e^{-2as})}$.

11. $\frac{1 - e^{-\pi s}}{s^2(1 + e^{-\pi s})}$.

12. $\frac{e^{-as} + e^{-2as} - 2e^{-3as}}{s(1 - e^{-3as})} = \frac{e^{-as}(1 + 2e^{-as})}{s(1 + e^{-as} + e^{-2as})}$.

13. $\left[\frac{1}{s} + \frac{2e^{-\pi s/2}}{s^2(1 - e^{-\pi s})}\right]\left(\frac{s^2}{1 + s^2}\right)$.

14. $Y(s) = \frac{1}{s[(s+2)^2 + 1]}\left[\frac{2}{1 + e^{-\pi s}} - 1\right] = \frac{1}{s[(s+2)^2 + 1]}[1 - 2e^{-\pi s} + 2e^{-2\pi s} - \ldots]$

$$\mathcal{L}^{-1}\left[\frac{1}{s\{(s+2)^2 + 1\}}\right] = \int_0^t e^{-2t}\sin t\, dt = \frac{1}{5}[1 - e^{-2t}(2\sin t + \cos t)] = \frac{1}{5}[1 - g(t)]$$

$$y(t) = \frac{1}{5}[1 - g(t)]\, u_0(t) - \frac{2}{5}[1 - g(t - \pi)]u_\pi(t) + \frac{2}{5}[1 - g(t - 2\pi)]u_{2\pi}(t) - \ldots.$$

15. $Y(s) = \frac{1}{(s^2+1)(s^2+4)}[1 + e^{-\pi s} + e^{-2\pi s} + \ldots]$

$y(t) = g(t)\, u_0(t) + g(t - \pi)\, u_\pi(t) + g(t - 2\pi)\, u_{2\pi}(t) + \ldots$, where $g(t) = \frac{1}{6}(2\sin t - \sin 2t)$.

16. $Y(s) = \frac{s}{s^2 + 9} + \frac{s}{(s^2+9)(s^2+1)(1 - e^{-\pi s})} = \frac{s}{s^2+9} + \frac{s}{(s^2+9)(s^2+1)}(1 + e^{-\pi s} + e^{-2\pi s} + \ldots)$

$y(t) = (\cos 3t)\, u_0(t) + g(t)u_0(t) + g(t - \pi)\, u_\pi(t) + g(t - 2\pi)\, u_{2\pi}(t) + \ldots$

where $g(t) = (\cos t - \cos 3t)/8$. Since $g(t - \pi) = -g(t)$, $g(t - 2\pi) = g(t)$, etc.

we have $y(t) = \cos 3t\, u_0(t) + g(t)\,[u_0(t) - u_\pi(t) + u_{2\pi}(t) - u_{3\pi}(t) + \ldots]$.

17. $Y(s) = \frac{1}{s^2 + 6s + 10} + \frac{1}{s(s^2 + 6s + 10)(1 + e^{-\pi s})}$

$$= \frac{1}{s^2 + 6s + 10} + \frac{1}{s(s^2 + 6s + 10)}(1 - e^{-\pi s} + e^{-2\pi s} - \ldots)$$

$$y(t) = e^{-3t} \sin t\, u_0(t) + \frac{1}{10} [u_0(t) - u_\pi(t) + u_{2\pi}(t) - \ldots] + g(t)[u_0(t) + e^{3\pi} u_\pi(t) + e^{6\pi} u_{2\pi}(t) + \ldots]$$

where $g(t) = -e^{-3t}(3 \sin t + \cos t)/10$.

18. $$Y(s) = \frac{1}{(s+1)(s+2)} \left[\frac{1}{s^2} - \frac{ae^{-as}}{s(1-e^{-as})} \right]$$

$$y(t) = g_1(t) u_0(t) - a \left[\left\{ \frac{1}{2} + g_2(t-a) \right\} u_a(t) + \left\{ \frac{1}{2} + g_2(t-2a) \right\} u_{2a}(t) + \ldots \right]$$

where $g_1(t) = -\frac{3}{4} + \frac{t}{2} + 2e^{-t} - \frac{5}{4} e^{-2t}$, $g_2(t) = \frac{1}{2} e^{-2t} - e^{-t}$.

19. $$Y(s) = \frac{1}{s^2(s+1)(s+3)} \left(\frac{1-e^{-\pi s}}{1+e^{-\pi s}} \right)$$

$$y(t) = \left[-\frac{4}{9} + g(t) \right] u_0(t) - 2 \left[\left\{ -\frac{4}{9} + g(t-\pi) \right\} u_\pi(t) - \left\{ -\frac{4}{9} + g(t-2\pi) \right\} u_{2\pi}(t) + \ldots \right]$$

$$g(t) = \frac{1}{3} t + \frac{1}{2} e^{-t} - \frac{1}{18} e^{-3t}.$$

20. $$Y(s) = -\frac{e^{-\pi s}}{(s^2+1)^2(1-e^{-\pi s})}$$

$$y(t) = g(t-\pi)\, u_\pi(t) + g(t-2\pi)\, u_{2\pi}(t) + \ldots; \quad g(t) = \frac{1}{2} (t \cos t - \sin t).$$